Uncontrollable Networks for Laplacian Leader-Follower Dynamics

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Uncontrollable Networks for Laplacian Leader-Follower Dynamics

Thesis directed by Prof. Sean O'Rourke

In this paper, we study characterizations of controllability for Laplacian leader-follower dynamics. Our discussion relies on the classification of graphs (networks) into three controllability classes: essentially controllable, completely uncontrollable, and conditionally controllable. In particular, this paper will show characteristics of graphs and their Laplacian matrices which give rise to complete uncontrollability. The controllability classes mentioned require the additional specification of the control vectors; here, we focus on the set of binary control vectors. We show that any Laplacian matrix with repeated eigenvalues will always have completely uncontrollable Laplacian leader-follower dynamics. This fact will allow us to make several deductions regarding the controllability properties of certain graph structures. Specifically, we prove that every circulant graph will have completely uncontrollable Laplacian leader-follower dynamics. We also show that for any biregular graph, if the difference in size between the two vertex sets is greater than one, then the Laplacian leader-follower dynamics are completely uncontrollable. In addition, we prove a similar bound for bipartite graphs in general.

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Chapter 1

Preliminaries

1.1 Introduction

Networks are systems which consist of *agents* that have connections with certain other agents in the network. Networks appear in a large number of applications, such as social and economic scenarios [8]. Although this topic has been studied for decades, it has gained attention in recent years; there is an interest in knowing what fundamental properties of networks will cause them to be controllable or not. The controllability problem called *leader-follower dynamics* has in particular received considerable attention [2, 3, 5, 11–14, 16, 17]. Given any graph (network), this study investigates the controllability properties of the leader-follower dynamics associated with the Laplacian matrix of the graph.

In this initial chapter, we will discuss some concepts from graph theory and control theory which are required to understand Laplacian leader-follower dynamics. After this definition, we present some features of linear algebra that will be required later in the paper.

1.2 Notation

Throughout this paper, upper case symbols like E and Y will refer to graphs or sets. If two graphs X and Y are isomorphic, this relationship is denoted with $X \cong Y$. The cardinality of a set S is denoted $|S|$. Upper case bold characters like $\mathbf{A}$ and $\mathbf{L}$ will refer to matrices, while lower case bold characters like $\mathbf{v}$ and $\mathbf{x}$ will refer to vectors (any $n \times 1$ matrix is indifferent from a vector). The symbol $\|\mathbf{v}\|$ denotes the Euclidean norm of vector $\mathbf{v}$. We use $\mathbf{I}_n$ to represent the $n \times n$ identity matrix, $\mathbf{0}_n$ for the vector in $\mathbb{R}^n$ with zero for all entries, and $\mathbf{1}_n$ for the vector in $\mathbb{R}^n$ with one for all entries, and will often drop the subscript when the dimension is clear. The vector $\boldsymbol{e}_j$ will represent the standard basis vector in $\mathbb{R}^n$ with the nonzero term in the j-th component. We will write the dot/inner product of two vectors $\mathbf{x}$ and $\mathbf{y}$ as $\mathbf{x} \cdot \mathbf{y}$ or $\mathbf{x}^{\mathrm{T}}\mathbf{y}$ interchangeably.

Many of the results that are found in Chapters 3 and 4 rest on theorems from linear algebra, which are discussed near the end of this chapter. To that end, we dedicate the remainder of this section to linear algebra notation/definitions. Let V be a finite-dimensional vector space. If every vector in V can be written as a unique linear combination of vectors in the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, where all the $\mathbf{v}_i$ are non-zero, linearly independent vectors, then we call $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ a **basis** for V. If the basis has the property that $\mathbf{v}_i^{\mathrm{T}}\mathbf{v}_j = 0$ for $i \neq j$ and $\|\mathbf{v}_i\| = 1$ for all $i \in \{1, \ldots, n\}$, then we say $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is an **orthonormal** basis for V. Next, define $\dim V = n$, called the **dimension** of V, which is the (unique) number of vectors in a basis for V. The symbol $\operatorname{span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ represents the vector space which is the set of all linear combinations of $\mathbf{a}_1 \ldots \mathbf{a}_n$. Now let $\mathbf{A}$ be an $n \times m$ matrix. The **column space** of $\mathbf{A}$, denoted $\operatorname{col}\mathbf{A}$, is the space formed by all linear combinations of the columns of $\mathbf{A}$. We call the dimension of this space the **rank** of $\mathbf{A}$, so that $\operatorname{rank}\mathbf{A} = \dim(\operatorname{col}\mathbf{A})$. Finally, we will also use the **kernel** of $\mathbf{A}$, denoted $\ker\mathbf{A}$, which is the space formed by taking linear combinations of the vectors $\mathbf{v}$ such that $\mathbf{A}\mathbf{v} = \mathbf{0}$.

1.3 Graph Theory

A **graph** X consists of a vertex set $V(X)$ and an edge set $E(X)$, where an edge is an unordered pair of distinct vertices of X. For the purposes of this study, we always label the vertices of graphs with positive integers, so that for any graph X with n nodes, $V(X) = \{1, 2, \ldots, n\}$ for positive integer n. If i and j are connected by an edge in the graph X, then the pair $\{i, j\} \in E(X)$, and we say that i and j are **adjacent**, or that j is a **neighbor** of i (and vice versa). The **degree** or **valency** d_i of a vertex i is the number of edges in $E(X)$ which are connected to i, i.e., the valency is the number of neighbors a vertex has. If every vertex in graph X has degree k, then we say X is k**-regular**. Notice that $\{i, j\} \in E(X) \iff \{j, i\} \in E(X)$ when $i \neq j$ and $\{i, j\} \notin E(X)$

when $i = j$ for all $i, j \in V(X)$. We would like to clarify that there can be no edges of the form $\{i, i\}$, where i is a vertex, i.e., there are no "loops" in the graph. Also, there are no directed edges such that $\{i, j\} \in E(X)$ and $\{j, i\} \notin E(X)$ for $i, j \in V(X)$, i.e., there are no "arrows" in the graph. Figure 1.1 depicts a graph as described, and Example 1.3.1 gives the vertex set and edge set of this graph. Finally, a **subgraph** Y of X has the property that $V(Y) \subseteq V(X)$ and $E(Y) \subseteq E(X)$, where if $i, j \in V(Y)$, then $\{i, j\} \in E(Y)$ if and only if $\{i, j\} \in E(X)$. The following definitions articulate how we use square matrices to represent the structure of graphs.

Definition 1.3.1 (Adjacency Matrix)**.** *Given a graph X, its **adjacency matrix**, denoted $\boldsymbol{A}(X)$, is a symmetric matrix whose entries are either 0 or 1, where the rows and columns are indexed by the vertices $i \in V(X)$. We define the entries of the adjacency matrix $\boldsymbol{A} = \boldsymbol{A}(X)$ as*

$$\boldsymbol{A}_{ij} = \begin{cases} 1 & \text{if } \{i, j\} \in E(X) \\ 0 & \text{if } \{i, j\} \notin E(X) \end{cases}. \tag{1.1}$$

From this definition it is clear that $\mathbf{A}$ is symmetric because $\{i, j\} \in E(X)$ if and only if $\{j, i\} \in E(X) \implies \mathbf{A}_{ij} = \mathbf{A}_{ji}$ for all $i, j \in V(X)$. This is an important fact of the adjacency matrix which is a cornerstone to many of this paper's ideas.

Definition 1.3.2 (Laplacian Matrix)**.** *The **degree matrix** $\boldsymbol{D}(X)$ of a graph X is the diagonal matrix with rows and columns indexed by the vertices of X, whose entries are the valency of each vertex $i \in V(X)$. Explicitly, the degree matrix $\boldsymbol{D} = \boldsymbol{D}(X)$ has entries*

$$\boldsymbol{D}_{ij} = \begin{cases} d_i & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}, \tag{1.2}$$

*where d_i is the degree of vertex i. The **Laplacian matrix** of X, denoted $\boldsymbol{L}(X)$, is defined as*

$$\boldsymbol{L}(X) = \boldsymbol{D}(X) - \boldsymbol{A}(X). \tag{1.3}$$

In words, the Laplacian matrix $\mathbf{L} = \mathbf{L}(X)$ for some graph X has the valency of vertex $i \in V(X)$ for diagonal entries $\mathbf{L}_{ii}$, -1 for entry $\mathbf{L}_{ij}$ if i and j are neighbors ($i \neq j$), and 0 everywhere else. Example 1.3.1 gives the Laplacian matrix of the graph depicted in Figure 1.1.

Example 1.3.1. *The vertex set, edge set, and Laplacian matrix of the graph in Figure 1.1 are*

$$V = \{1, 2, 3, 4, 5, 6\}, \qquad E = \{\{1,4\}, \{2,3\}, \{2,4\}, \{2,5\}, \{3,4\}, \{4,5\}, \{4,6\}, \{5,6\}\},$$

and

$$\boldsymbol{L} = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 3 & -1 & -1 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ -1 & -1 & -1 & 5 & -1 & -1 \\ 0 & -1 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{pmatrix}.$$

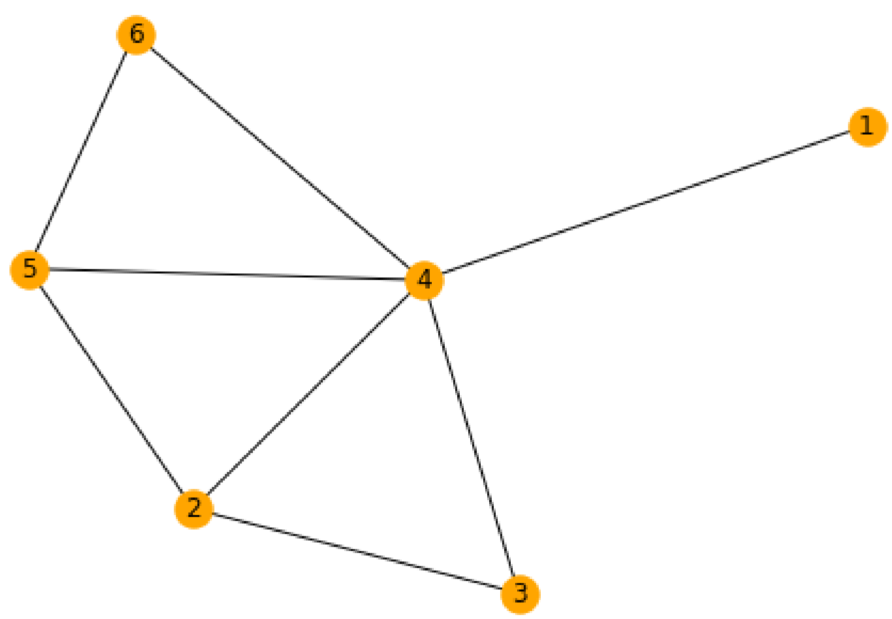

Figure 1.1: The graph with six nodes described in Example 1.3.1.

Next, we want to make a statement concerning graphs which are not equivalent, yet whose structure is equivalent. Even though we have defined graphs as a vertex set and edge set, there is no structural difference in the graphs if the vertices can be relabeled to be made equivalent, as is the case in Figure 1.2. Here we formally define this concept of isomorphic graphs.

Definition 1.3.3 (Isomorphic Graphs)**.** *Let X and Y be two graphs on n vertices, so that $V(X) = V(Y)$. Consider a bijection $\phi : V(X) \to V(Y)$ such that vertex i is a neighbor of vertex j in X if and only if vertex $\phi(i)$ is a neighbor of vertex $\phi(j)$ in Y. If such a map exists, we say X and Y are* ***isomorphic****, and write $X \cong Y$.*

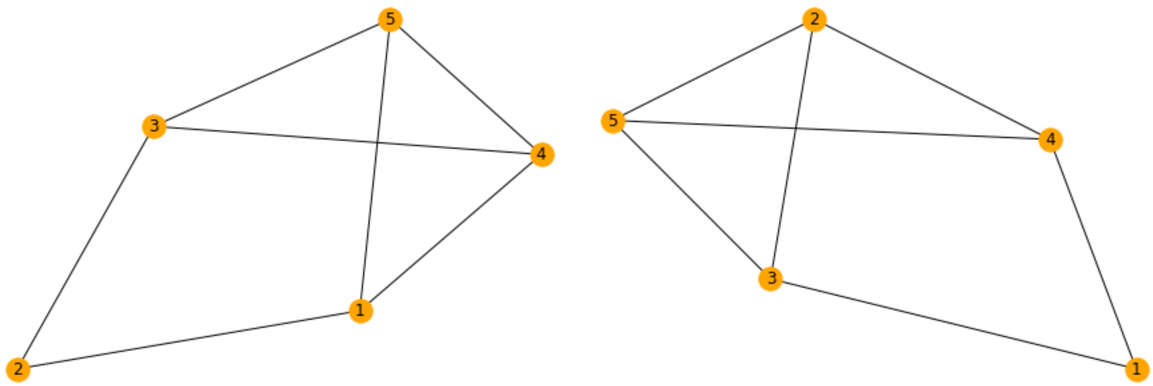

Figure 1.2: Two distinct yet isomorphic graphs.

Notice that even though these graphs are not equivalent, the map

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 3 & 1 & 5 \end{pmatrix}$$

permutes the vertices of the graph on the left so that the resulting graph is exactly the graph on the right.

With this understanding of graphs and the Laplacian matrix that describes them, we are halfway to defining Laplacian leader-follower dynamics. The other half requires us to step away from graph theory momentarily, and shift our attention towards linear systems of differential equations, with a perspective from control theory.

1.4 Control Theory

This study is fundamentally motivated by the continuous-time state-space linear system given by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \tag{1.4}$$

where $t \in \mathbb{R}$, $\mathbf{x} : \mathbb{R} \to \mathbb{R}^n$ is the state of the system, $\mathbf{A} \in \mathbb{R}^{n\times n}$ captures the dynamics of the system, $\mathbf{B} \in \mathbb{R}^{n\times k}$ is a **control matrix**, $\mathbf{u} : \mathbb{R} \to \mathbb{R}^k$ is the piecewise-continuous input to the system, and $\dot{\mathbf{x}}(t)$ is equivalent to $\frac{d\mathbf{x}}{dt}$. The text by Hespanha [7] gives a much more general definition of linear systems, and explores the topic rigorously. A real world example of this kind of system is aircraft autopilot; $\mathbf{x}(t)$ would represent the state of the aircraft at time t, having components which correspond to velocity, pitch, altitude etc.... The matrix $\mathbf{A}$ would then dictate how the system evolves on its own, without control (i.e. $\mathbf{u}(t) = \mathbf{0}$). For the airplane example, the matrix $\mathbf{A}$ would normally steer an unattended, inactive aircraft to fall to the Earth and crash. The control matrix $\mathbf{B}$ would then relate to the response of the aircraft to certain inputs like power to the engine and position of the flaps/tail rudder, reflected by $\mathbf{u}(t)$. This function $\mathbf{u}(t)$ is the variable control, which can change the evolution of the system. The autopilot of our aircraft would be calculating what $\mathbf{u}(t)$ should be, so that for some $0 \leq t_0 < t_1$, the system evolves from the current state $\mathbf{x}(t_0)$ to the desired state $\mathbf{x}(t_1)$ (this airplane example is an over-simplification, but is useful for understanding what the different objects in (1.4) represent). Now that we have a concrete definition of the continuous-time state-space linear system, we can discuss what we mean by a "controllable" system.

Definition 1.4.1 (Controllable System)**.** *Consider the system in* (1.4)*. Given two times* $0 \leq t_0 < t_1$*, let the* ***reachability*** *of the system be the space of all states* $\boldsymbol{y} \in \mathbb{R}^n$ *for which there exists some* $\boldsymbol{u}(t)$ *that transfers* $\boldsymbol{x}$ *from* $\boldsymbol{x}(t_0) = \boldsymbol{0}$ *to* $\boldsymbol{x}(t_1) = \boldsymbol{y}$*. If the reachability is all of* $\mathbb{R}^n$*, then we say the system is* ***controllable****, or equivalently, that the pair* $(\boldsymbol{A}, \boldsymbol{B})$ *is controllable.*

The next two lemma are useful tests for determining if a given system is controllable. The first is Kalman's rank condition $\big($see [9], equation (2.8)$\big)$. A similar definition is used in [7, 13].

Lemma 1.4.1 (Rank of Controllability Matrix)**.** *Consider the system in* (1.4)*. Define the* ***controllability matrix $\boldsymbol{C}$*** *of the system as*

$$\boldsymbol{C} = \begin{pmatrix} \boldsymbol{B} & \boldsymbol{AB} & \dots & \boldsymbol{A}^{n-1}\boldsymbol{B} \end{pmatrix}. \tag{1.5}$$

Then the pair $(\boldsymbol{A}, \boldsymbol{B})$ is controllable if and only if

$$rank\ \boldsymbol{C} = n. \tag{1.6}$$

The proof of this lemma is beyond the scope of the topics central to this paper, but full details are provided in Hespanha's text, Lectures 11 and 12 [7]. Additionally, for more detail on the relation between control theory and graph theory, see [16].

If the control matrix $\mathbf{B}$ is an $n \times 1$ matrix, then we think of it as a vector, and denote it with $\mathbf{b}$. With this distinction, we have the following lemma, which is a principle aspect of control theory, and is used in [11, 13].

Lemma 1.4.2 (Popov-Belevitch-Hautus (PBH) test for controllability)**.** *Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ be a real symmetric matrix, and let $\boldsymbol{b} \in \mathbb{R}^n$. Then the pair $(\boldsymbol{A}, \boldsymbol{b})$ is controllable if and only if for all eigenvectors $\boldsymbol{v}$ of $\boldsymbol{A}$, $\boldsymbol{v}^T\boldsymbol{b} \neq 0$.*

This controllability criteria was first discovered by G. Strang in 1963 only for the case of diagonalizable systems. The test was generalized (independently) by V. Popov in 1966, V. Belevitch in 1968, and M. Hautus in 1969 [10].

Now that we have defined the idea of linear systems and controllability, we are ready to describe Laplacian leader-follower dynamics.

1.5 Problem Statement

This project studies the system in (1.4) under the two following conditions:

(1) The control matrix $\mathbf{B}$ is restricted to the set of n-dimensional binary vectors (i.e., the vectors with either zero or one for each entry), except for the all-zeros vector and all-ones vector (these turn out to be trivial cases, see Theorem 2.2.1), written as $\{0,1\}^n \backslash \{\mathbf{0}, \mathbf{1}\}$. This condition implies that $\mathbf{u}(t) = u(t) : \mathbb{R} \to \mathbb{R}$. Additionally, we will refer to this as the **control vector**, and write it as $\mathbf{b} \in \{0,1\}^n \backslash \{\mathbf{0}, \mathbf{1}\}$.

(2) The matrix $\mathbf{A}$ which captures the dynamics of the system is the Laplacian matrix of a graph.

We formalize these conditions in the following definition.

Definition 1.5.1 (Laplacian Leader-Follower Dynamics)**.** *Given a graph X with Laplacian matrix $\boldsymbol{L} = \boldsymbol{L}(X)$, consider the single-input time-independent linear control system*

$$\dot{\boldsymbol{x}}(t) = -\boldsymbol{L}\boldsymbol{x}(t) + \boldsymbol{b}u(t), \tag{1.7}$$

where $t \in \mathbb{R}$, $\boldsymbol{x} : \mathbb{R} \to \mathbb{R}^n$ is the state of the system, $\boldsymbol{b} \in \{0,1\}^n \backslash \{\boldsymbol{0}, \boldsymbol{1}\}$ is a control vector, and $u : \mathbb{R} \to \mathbb{R}$ is the input. We refer to (1.7) *as the* ***Laplacian leader-follower dynamics*** *(or* ***LLFD****) of X.*

We may refer to this notion as the “LLFD of X” or the “LLFD of $\mathbf{L}(X)$” interchangeably, since a graph is completely described by its unique Laplacian matrix. A real-world system that relates to Laplacian leader-follower dynamics is a drone swarm. Suppose there are n drones, represented by the vertices of a graph X. Some of the drones are able to communicate with each other; these connections are represented by the edges of X. Also, some of the drones (vertices) are set to be **leaders**, and the rest are set to be **followers**. This classification is reflected in the control vector $\mathbf{b} \in \{0,1\}^n \backslash \{\mathbf{0}, \mathbf{1}\}$, where the i-th component of $\mathbf{b}$ is one if drone i is a leader, and zero if the drone is a follower. As for the airplane example with the general linear system, this drone swarm example is also not a rigorous explanation, but a helpful example to understand (1.7).

As mentioned earlier, two important control vectors in $\{0,1\}^n$ are $\mathbf{0}$ and $\mathbf{1}$, which correspond to the network having no leaders and exclusively leaders, respectively. Intuitively, one might think that either of these networks would be uncontrollable. Indeed, the pairs $(\mathbf{L}, \mathbf{0})$ and $(\mathbf{L}, \mathbf{1})$ are uncontrollable for any Laplacian $\mathbf{L}$, the proof of which is postponed until Chapter 2 (Theorem 2.2.1).

The LLFD of a graph X will belong to one of three controllablility classes:

Definition 1.5.2 (Controllability Classes of LLFD)**.** *Consider the LLFD of a graph X with n vertices and Laplacian $\boldsymbol{L} = \boldsymbol{L}(X)$. Define $B = \{0,1\}^n \backslash \{\boldsymbol{0}, \boldsymbol{1}\}$. Then exactly one of the following is true:*

*(1) The pair $(\boldsymbol{L}, \boldsymbol{b})$ is controllable for all $\boldsymbol{b} \in B$. If this is the case, we say the LLFD of X are **essentially controllable**.*

*(2) The pair $(\boldsymbol{L}, \boldsymbol{b})$ is uncontrollable for all $\boldsymbol{b} \in B$. If this is the case, we say the LLFD of X are **completely uncontrollable**.*

*(3) If (1) and (2) are not true, then we say the LLFD of X are **conditionally controllable**, that is, the pair $(\boldsymbol{L}, \boldsymbol{b})$ is controllable for $\boldsymbol{b}$ in some non-empty proper subset of B, and uncontrollable for $\boldsymbol{b}$ in the complement set.*

The **Erdős-Rényi random graph** $G(n,p)$ with n vertices and edge density p is a graph with n nodes such that any unordered pair of vertices is an edge of $G(n,p)$ with probability p. When $p = 1/2$, we think of the graph $G(n, 1/2)$ as a random selection from a uniform distribution of all graphs on n nodes. It was shown in Theorem 1.4 and Theorem 1.5 of [13] that if $\mathbf{A}$ is the adjacency matrix of $G(n, 1/2)$, then for all $\alpha > 0$, there exists $C > 0$ such that the probability that the pair $(\mathbf{A}, \mathbf{1})$ is controllable is $\geq 1 - Cn^{-\alpha}$ $\big($note that although we claimed $(\mathbf{L}, \mathbf{1})$ is always uncontrollable, the same is not true for $(\mathbf{A}, \mathbf{1})\big)$. Continuing that idea in this study, we investigate whether a similar pattern appears for Laplacian leader-follower dynamics, where we consider the

Laplacian matrix $\mathbf{L}$ instead of the adjacency matrix $\mathbf{A}$. We are also working with the pair $(\mathbf{L}, \mathbf{b})$, where $\mathbf{b} \in \{0,1\}^n \backslash \{\mathbf{0}, \mathbf{1}\}$. Initially, we suspected that the LLFD of $G(n, 1/2)$ would more frequently be essentially controllable with growing n. Our simulation suggests that, indeed, as $n \to \infty$, the probability that the LLFD of X are essentially controllable goes to one, and the probability that the LLFD of X are conditionally controllable/completely uncontrollable goes to zero, as Table 1.1 and Figure 1.3 illustrate.

n	CU	CC	EC
6	0.406	0.501	0.093
7	0.206	0.596	0.198
8	0.145	0.535	0.32
9	0.082	0.419	0.499
10	0.056	0.351	0.593
11	0.025	0.191	0.784
12	0.014	0.185	0.801
13	0.006	0.078	0.916
14	0.004	0.08	0.916
15	0.001	0.035	0.964

Table 1.1: Proportion of each controllability class for graphs with six to fifteen vertices.

This data is plotted in Figure 1.3. For each vertex set size n, our simulation generated one thousand Laplacian matrices of Erdős-Rényi random graphs with edge density $p = 1/2$, not up to isomorphism, and including repetitions (see Appendix B). For the controllability classes completely uncontrollable (CU), conditionally controllable (CC), and essentially controllable (EC), the table presents the proportion of the one thousand random graphs that belong to each class.

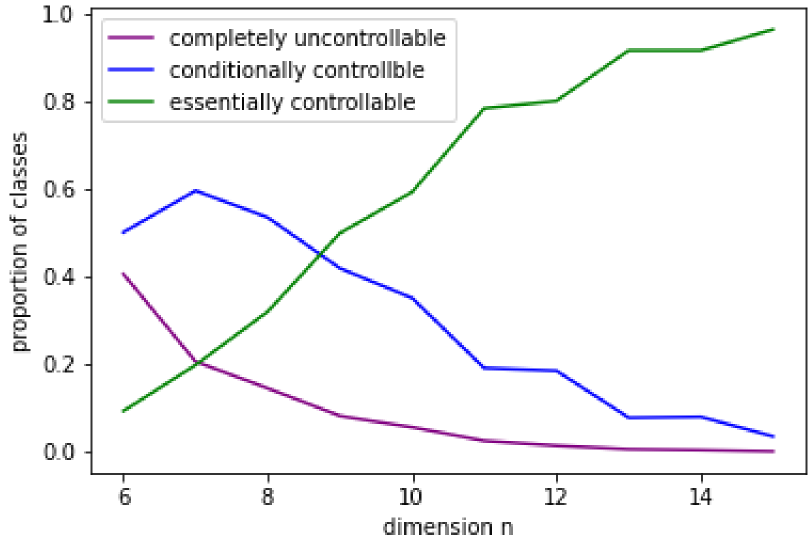

Figure 1.3: Plot of the simulation data in Table 1.1.

Notice that the class of essentially controllable systems (green) seems to quickly approaches a proportion of one.

1.6 Isomorphic Graphs

This section is brief, yet important for the sake of generality. Given two isomorphic graphs, we would expect that their LLFD belong to the same controllability classes. Indeed, this fact is true, and we provide proof using the PBH test.

Lemma 1.6.1 (Isomorphic Graphs Belong to the Same Controllability Class). *Let X and Y be two distinct graphs which are isomorphic. Then the LLFD of X and Y belong to the same controllability class.*

Proof. Let X and Y be two isomorphic graphs. There exists the map $\pi : V(X) \to V(Y)$ which permutes the vertices of X such that the resulting graph is Y. Define the **permutation matrix** as

$$\mathbf{P}_{xy} = \begin{cases} 1 & \text{if } \pi(x) = y \\ 0 & \text{else} \end{cases}, \text{ for } x \in V(X) \text{ and } y \in V(Y). \tag{1.8}$$

This $\mathbf{P}$ permutes the rows and columns of $\mathbf{L}(X)$ so that the resulting matrix is $\mathbf{L}(Y)$, i.e.

$$\mathbf{P}^{-1}\mathbf{L}(X)\mathbf{P} = \mathbf{L}(Y). \tag{1.9}$$

Also note that $\mathbf{P}^{\mathrm{T}} = \mathbf{P}^{-1}$. Let $\mathbf{v}$ be an eigenvector of $\mathbf{L}(X)$ with eigenvalue λ. Then

$$\begin{aligned} \mathbf{L}(Y)(\mathbf{P}^{-1}\mathbf{v}) &= \left(\mathbf{P}^{-1}\mathbf{L}(X)\mathbf{P}\right)\left(\mathbf{P}^{-1}\mathbf{v}\right) \\ &= \mathbf{P}^{-1}\mathbf{L}(X)\left(\mathbf{P}\mathbf{P}^{-1}\right)\mathbf{v} \\ &= \mathbf{P}^{-1}\left(\mathbf{L}(X)\mathbf{v}\right) \\ &= \mathbf{P}^{-1}\lambda\mathbf{v} \\ &= \lambda(\mathbf{P}^{-1}\mathbf{v}). \end{aligned} \tag{1.10}$$

Hence if $\mathbf{v}$ is an eigenvector of $\mathbf{L}(X)$ with eigenvalue λ, then $\mathbf{P}^{-1}\mathbf{v}$ is an eigenvector of $\mathbf{L}(Y)$ with eigenvalue λ. Next, we will show that $\mathbf{v}^{\mathrm{T}}\mathbf{b} = 0 \iff (\mathbf{P}^{-1}\mathbf{v})^{\mathrm{T}}(\mathbf{P}^{-1}\mathbf{b}) = 0$. Observe that

$$\left(\mathbf{P}^{-1}\mathbf{v}\right)^{\mathrm{T}}\left(\mathbf{P}^{-1}\mathbf{b}\right) = \mathbf{v}^{\mathrm{T}}\mathbf{P}\mathbf{P}^{-1}\mathbf{b} = \mathbf{v}^{\mathrm{T}}\mathbf{b}, \tag{1.11}$$

where we used Lemma 1.7.1 to say that $\left(\mathbf{P}^{-1}\mathbf{v}\right)^{\mathrm{T}} = \mathbf{v}^{\mathrm{T}}\left(\mathbf{P}^{-1}\right)^{\mathrm{T}} = \mathbf{v}^{\mathrm{T}}\mathbf{P}$.
Since $\left(\mathbf{P}^{-1}\mathbf{v}\right)^{\mathrm{T}}\left(\mathbf{P}^{-1}\mathbf{b}\right) = \mathbf{v}^{\mathrm{T}}\mathbf{b}$, it must be the case that $\mathbf{v}^{\mathrm{T}}\mathbf{b} = 0 \iff (\mathbf{P}^{-1}\mathbf{v})^{\mathrm{T}}(\mathbf{P}^{-1}\mathbf{b}) = 0$. By the PBH test (Lemma 1.4.2), this means that the pair $(\mathbf{L}(X), \mathbf{b})$ is controllable if and only if the pair $(\mathbf{L}(Y), \mathbf{P}^{-1}\mathbf{b})$ is controllable. This means that X and Y have the same number of controllable pairs for $\mathbf{b} \in \{0,1\}^n \backslash \{\mathbf{0}, \mathbf{1}\}$, so they belong to the same controllability class. □

This immediately gives us a useful corollary:

Corollary 1.6.1. *Let γ denote the controllablity class of the graph X. Then for all graphs Y such that $Y \cong X$, Y also belongs to controllability class γ.*

1.7 Linear Algebra Tools

In this section we present some properties of linear algebra which we utilize later in the paper. These theorems along with their proofs can be found in most linear algebra textbooks. We offer some justification here, for more see [15] (for example).

Lemma 1.7.1. *For any matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ such that $\boldsymbol{AB}$ is defined,*

$$(\boldsymbol{AB})^T = \boldsymbol{B}^T\boldsymbol{A}^T. \tag{1.12}$$

Proof. If $\mathbf{A}_{ij} = a_{ij}$ and $\mathbf{B}_{ij} = b_{ij}$, then $(\mathbf{AB})_{ij}$ is given by

$$(\mathbf{AB})_{ij} = \sum_k a_{ik} b_{kj}. \tag{1.13}$$

Taking the transpose, we have

$$(\mathbf{AB})^{\mathrm{T}}_{ij} = (\mathbf{AB})_{ji} = \sum_k a_{jk} b_{ki}. \tag{1.14}$$

Now consider the matrix $\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$, with entries given by

$$\left(\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\right)_{ij} = \sum_k (\mathbf{B}^{\mathrm{T}})_{ik} (\mathbf{A}^{\mathrm{T}})_{kj} = \sum_k \mathbf{B}_{ki} \mathbf{A}_{jk} = \sum_k b_{ki} a_{jk}. \tag{1.15}$$

Thus by the equality of (1.14) and (1.15), $(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$. □

Lemma 1.7.2. *For any matrix $\boldsymbol{A}$:*

$$\operatorname{rank} \boldsymbol{A} = \operatorname{rank} \boldsymbol{A}^T. \tag{1.16}$$

See Theorem 1.12.11 and Theorem 1.13.9 in [15] for a proof of Lemma 1.7.2.

Lemma 1.7.3. *For any matrix $\boldsymbol{A}$ with n columns:*

$$\operatorname{rank} \boldsymbol{A} + \dim(\ker \boldsymbol{A}) = n. \tag{1.17}$$

This is an aspect of the famous *rank-nullity theorem*. See Theorem 11.1 in [7] for more about the rank-nullity theorem.

Lemma 1.7.4. *For any two $m \times n$ matrices $\boldsymbol{A}$ and $\boldsymbol{B}$:*

$$\operatorname{rank}(\boldsymbol{A} + \boldsymbol{B}) \leq \operatorname{rank} \boldsymbol{A} + \operatorname{rank} \boldsymbol{B}. \tag{1.18}$$

Proof. First observe that

$$\operatorname{rank}(\mathbf{A} + \mathbf{B}) = \dim(\operatorname{col}(\mathbf{A} + \mathbf{B})), \ \operatorname{rank} \mathbf{A} = \dim(\operatorname{col} \mathbf{A}), \ \text{and} \ \operatorname{rank} \mathbf{B} = \dim(\operatorname{col} \mathbf{B}). \tag{1.19}$$

Let $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ be the set of columns of $\mathbf{A}$, and $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ be the set of columns of $\mathbf{B}$. It follows that $\{\mathbf{a}_1 + \mathbf{b}_1, \mathbf{a}_2 + \mathbf{b}_2, \ldots, \mathbf{a}_n + \mathbf{b}_n\}$ is the set of columns of $\mathbf{A} + \mathbf{B}$. We can write

$$\begin{aligned} \operatorname{col}\mathbf{A} &= \operatorname{span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}, \\ \operatorname{col}\mathbf{B} &= \operatorname{span}\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}, \\ \operatorname{col}\mathbf{A+B} &= \operatorname{span}\{\mathbf{a}_1 + \mathbf{b}_1, \ldots, \mathbf{a}_n + \mathbf{b}_n\}. \end{aligned} \tag{1.20}$$

By the definition of spanning sets,

$$\operatorname{span}\{\mathbf{a}_1 + \mathbf{b}_1, \ldots, \mathbf{a}_n + \mathbf{b}_n\} \subseteq \operatorname{span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n, \mathbf{b}_1, \ldots, \mathbf{b}_n\}, \tag{1.21}$$

which means

$$\dim(\operatorname{col}(\mathbf{A} + \mathbf{B})) \leq \dim\left(\operatorname{span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n, \mathbf{b}_1, \ldots, \mathbf{b}_n\}\right) \leq \dim(\operatorname{col}\mathbf{A}) + \dim(\operatorname{col}\mathbf{B}), \tag{1.22}$$

or

$$\operatorname{rank}(\mathbf{A} + \mathbf{B}) \leq \operatorname{rank}\mathbf{A} + \operatorname{rank}\mathbf{B}. \tag{1.23}$$

□

Lemma 1.7.5. *Let $\boldsymbol{A}$ be a real square matrix. If* $\dim(\ker \boldsymbol{A}) > 1$*, then zero is a repeated eigenvalue of $\boldsymbol{A}$.*

Proof. If $\dim(\ker \mathbf{A}) > 1$, there there exist (at least) two linearly independent vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $\mathbf{A}\mathbf{v}_1 = \mathbf{0}\mathbf{v}_1 = \mathbf{0} = \mathbf{0}\mathbf{v}_2 = \mathbf{A}\mathbf{v}_2$. This means that $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of $\mathbf{A}$ with eigenvalue 0. Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, we know that the eigenvalue 0 of $\mathbf{A}$ must have a multiplicity of at least two. □

Lemma 1.7.6. *If a matrix is Hermitian, then all of its eigenvalues are real.*

Proof. Let $\mathbf{A}$ be an $n \times n$ Hermitian matrix, and suppose $\lambda \in \mathbb{C}$ is an eigenvalue of $\mathbf{A}$, with corresponding eigenvector $\mathbf{v} \in \mathbb{C}^n$ ($\mathbf{v} \neq \mathbf{0}$). We will let $\overline{\lambda}$ and $\overline{\mathbf{v}}$ represent the respective conjugates of λ and $\mathbf{v}$, and denote the complex transpose with the asterisk *. Consider the expression, $\mathbf{v}^*\mathbf{A}\mathbf{v}$, which can be solved in the two following ways:

$$\mathbf{v}^*\mathbf{A}\mathbf{v} = \mathbf{v}^*(\mathbf{A}\mathbf{v}) = \mathbf{v}^*\lambda\mathbf{v} = \lambda(\mathbf{v}^*\mathbf{v}). \tag{1.24}$$

$$\mathbf{v}^*\mathbf{A}\mathbf{v} = \mathbf{v}^*\mathbf{A}^*\mathbf{v} = (\mathbf{A}\mathbf{v})^*\mathbf{v} = (\lambda\mathbf{v})^*\mathbf{v} = \overline{\lambda}(\mathbf{v}^*\mathbf{v}). \tag{1.25}$$

We used the fact that $\mathbf{A} = \mathbf{A}^*$ ($\mathbf{A}$ is Hermitian) and $\mathbf{v}^*\mathbf{A}^* = (\mathbf{A}\mathbf{v})^*$ (Lemma 1.7.1) in (1.25). Combining (1.24) and (1.25), we obtain

$$\overline{\lambda}(\mathbf{v}^*\mathbf{v}) = \lambda(\mathbf{v}^*\mathbf{v}). \tag{1.26}$$

Since $\mathbf{v}$ is not the all-zeros vector, $\mathbf{v}^*\mathbf{v} \neq 0$, and the only solution to (1.26) is $\lambda = \overline{\lambda}$, which means λ must be real. □

Lemma 1.7.7 (Shifting eigenvalues)**.** *Let $\boldsymbol{A}$ be an $n \times n$ matrix with eigenvector $\boldsymbol{v}$ with corresponding eigenvalue λ. Then for any scalar k, the matrix*

$$\boldsymbol{A} - k\boldsymbol{I} \tag{1.27}$$

has eigenvector $\boldsymbol{v}$ with corresponding eigenvalue $\lambda - k$.

Proof. Notice that

$$(\mathbf{A} - k\mathbf{I})\,\mathbf{v} = \mathbf{A}\mathbf{v} - k\mathbf{I}\mathbf{v} = \lambda\mathbf{v} - k\mathbf{v} = (\lambda - k)\mathbf{v}. \tag{1.28}$$

Hence, $\mathbf{v}$ is an eigenvector of $\mathbf{A} - k\mathbf{I}$ with corresponding eigenvalue $\lambda - k$. □

1.8 Overview of Paper

Now that we have used graph theory and control theory to define Laplacian leader-follower dynamics, along with some linear algebra tools for later, we move on to the results of the study. In Chapter 2, we discuss the controllability properties of some trivial small graphs, and then make some deductions concerning the controllability of LLFD based on the structure of the Laplacian matrix alone. Importantly, we prove a useful theorem pertaining to whether or not the eigenvalues of a Laplacian matrix are distinct, and use it to prove that disconnected graphs are always uncontrollable. In Chapter 3, we present findings on the specific class of circulant graphs, whose cyclic structure gives rise to uncontrollability. Similarly, in Chapter 4 we discuss bipartite graphs, making some general remarks about biregular graphs. Additionally, the appendix contains some discussion of the simulations developed by the author of this study and what potential there is for future research on the topic.

Chapter 2

Small Graphs and General Results for the Laplacian Matrix

In this chapter we present some initial results concerning the structure of the Laplacian matrix, proving several facts which apply to every possible graph. First, we will make some remarks regarding a set of very small graphs.

2.1 Graphs with 0, 1, or 2 Vertices

The graphs on $n = 0, 1, 2$ vertices behave differently than graphs with $n \geq 3$ vertices, so we consider the former separately in this section. The graph with zero vertices is not of interest in the subject of control theory, as there are no agents to be controlled. With one vertex, we have only one graph that can be constructed: the one with one vertex and no edges. This graph is also not of interest.

On $n = 2$ nodes, there are two graphs: one with an edge, and one without an edge. We examine both in the following examples:

Example 2.1.1. *The Laplacian matrix $\boldsymbol{L} = \boldsymbol{L}(X)$ of the graph X with two nodes and one edge is $\boldsymbol{L} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. By inspection, the eigenvectors of $\boldsymbol{L}$ are $[1\ 1]^T$ and $[1\ -1]^T$, with eigenvalues 0 and 2, respectively. For $n = 2$, the set $\{0,1\}^n \backslash \{\boldsymbol{0}, \boldsymbol{1}\}$ only contains $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$. We have*

$$\begin{aligned} [1\ 1]^T \boldsymbol{e}_1 = [1\ 1]^T \boldsymbol{e}_2 = 1 \neq 0, \\ [1\ -1]^T \boldsymbol{e}_1 = 1 \neq 0, \ \textit{and} \ \ [1\ -1]^T \boldsymbol{e}_2 = -1 \neq 0. \end{aligned} \tag{2.1}$$

Since $\mathbf{v}^T\boldsymbol{b} \neq 0$ *for all eigenvectors* $\boldsymbol{v}$ *and all control vectors* $\boldsymbol{b}$*, the LLFD of X are essentially controllable by the PBH test (Lemma 1.4.2).*

Example 2.1.2. *The Laplacian matrix* $\boldsymbol{L} = \boldsymbol{L}(Y)$ *of the graph Y with two nodes and no edges is* $\boldsymbol{L} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$*. The eigenvectors of this matrix are all of* $\mathbb{R}^2\backslash\{\boldsymbol{0}\}$*, with eigenvalue 0. For* $n = 2$*,* $\{0,1\}^n\backslash\{\boldsymbol{0}, \boldsymbol{1}\} = \{\boldsymbol{e}_1,\ \boldsymbol{e}_2\}$*. Since* $\boldsymbol{e}_1, \boldsymbol{e}_2 \in \mathbb{R}^2$*,* $\boldsymbol{e}_1, \boldsymbol{e}_2$ *are eigenvectors of* $\boldsymbol{L}$*.* $\boldsymbol{e}_1\boldsymbol{e}_2 = \boldsymbol{e}_2\boldsymbol{e}_1 = 0$*, so the pairs* $(\boldsymbol{L}, \boldsymbol{e}_1)$ *and* $(\boldsymbol{L}, \boldsymbol{e}_2)$ *are uncontrollable by Lemma 1.4.2 (PBH test), and the LLFD of X are completely uncontrollable.*

Later in this chapter, we will prove a theorem regarding repeated eigenvalues (Theorem 2.3.1), which makes the proof of Example 2.1.2 trivial.

2.2 The Trivial Control Vectors

As we have already mentioned, for any graph X, the pair $(\mathbf{L}(X), \mathbf{b})$ is uncontrollable if $\mathbf{b} = \mathbf{0}$ or $\mathbf{b} = \mathbf{1}$. This section will formalize the argument rigorously, and explain why $\mathbf{0}$ and $\mathbf{1}$ are referred to as the trivial control vectors.

Theorem 2.2.1. *Let X be some graph on* $n \geq 3$ *vertices with Laplacian matrix* $\boldsymbol{L} = \boldsymbol{L}(X)$*. Then the pairs* $(\boldsymbol{L}, \boldsymbol{0})$ *and* $(\boldsymbol{L}, \boldsymbol{1})$ *are uncontrollable.*

Proof. First, consider the pair $(\mathbf{L}, \mathbf{0})$. If $\mathbf{v}$ is an eigenvector of $\mathbf{L}$, then $\mathbf{v}^T\mathbf{0} = 0$, so the pair $(\mathbf{L}, \mathbf{0})$ is uncontrollable by the PBH test (Lemma 1.4.2).

Now we look at the pair $(\mathbf{L}, \mathbf{1})$. The Laplacian matrix $\mathbf{L}$ is defined in Chapter 1 as

$$\mathbf{L} = \mathbf{D} - \mathbf{A}, \tag{2.2}$$

where, as a reminder, $\mathbf{D}$ is the diagonal matrix whose i-th entry on the diagonal is the valency of vertex i, denoted d_i, and $\mathbf{A}$ is the adjacency matrix which, by definition, has d_i entries of 1 in the

i-th row. Matrix multiplication is distributive over addition, so we have

$$\mathbf{L1} = (\mathbf{D} - \mathbf{A})\mathbf{1} = \mathbf{D1} - \mathbf{A1} = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix} - \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix} = \mathbf{0}. \tag{2.3}$$

Since $\mathbf{L1} = \mathbf{0}$, $\mathbf{1}$ is an eigenvector of $\mathbf{L}$ with eigenvalue 0. The controllability matrix, defined in Lemma 1.4.1, is then

$$\mathbf{C} = \begin{pmatrix} \mathbf{1} & \mathbf{L1} & \dots & \mathbf{L}^{n-1}\mathbf{1} \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \dots & \mathbf{0} \end{pmatrix}, \tag{2.4}$$

since $\mathbf{L1} = \mathbf{0} \implies \mathbf{L}^{n-1}\mathbf{1} = \mathbf{0}$ for $n > 1$. The controllability matrix $\mathbf{C}$ clearly does not have full rank, so we can conclude that the pair $(\mathbf{L}, \mathbf{1})$ is uncontrollable by Kalman's rank condition (Lemma 1.4.1). $\square$

As we can now see, if we had defined the LLFD with a control vector that can be *any* binary vector $\mathbf{b} \in \{0,1\}^n$, then the class of essentially controllable graphs would be empty.

2.3 Repeated Eigenvalues

In order to make a general statement regarding Laplacian matrices with repeated eigenvalues, we prove the following lemma.

Lemma 2.3.1. *Let $\boldsymbol{A}$ be a real symmetric $n \times n$ matrix with eigenspace Λ such that* $\dim \Lambda \geq 2$ *(equivalently, $\boldsymbol{A}$ has repeated eigenvalues). Then for every $\boldsymbol{y} \in \mathbb{R}^n$ there exists some non-zero vector $\boldsymbol{r} \in \Lambda$ such that $\boldsymbol{y}^T\boldsymbol{r} = 0$.*

Proof. Since $\mathbf{A}$ is real symmetric, let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ be an orthonormal basis of Λ, so that $\Lambda = \operatorname{span}\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ ($\dim \Lambda = m \geq 2$). We will write the components of the basis vectors as $\mathbf{v}_i = [v_{i1}, v_{i2}, \dots, v_{in}]^{\mathrm{T}}$ (v_{ij} represents the j-th component of $\mathbf{v}_i$), $1 \leq i \leq m,\ 1 \leq j \leq n$. Let $c_1, \dots, c_m \in \mathbb{R}$ such that not all c_i are zero. We can write $\mathbf{r}$ as a linear combination of the basis vectors of Λ:

$$\mathbf{r} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \dots + c_m\boldsymbol{v}_m \in \operatorname{span}\{\mathbf{v}_1, \dots, \mathbf{v}_m\} = \Lambda \backslash \{\mathbf{0}\}. \tag{2.5}$$

By definition, $\mathbf{r}$ represents all vectors in Λ besides $\mathbf{0}$, so $\mathbf{r}$ is always an eigenvector of $\mathbf{A}$.

Now let $\mathbf{y} \in \mathbb{R}^n$. We have

$$\mathbf{y} \cdot \mathbf{r} = \mathbf{y} \cdot (c_1\mathbf{v}_1 + \cdots + c_m\mathbf{v}_m) = c_1(\mathbf{y} \cdot \mathbf{v}_1) + \cdots + c_m(\mathbf{y} \cdot \mathbf{v}_m). \tag{2.6}$$

Define $\beta_i = \mathbf{y} \cdot \mathbf{v}_i \in \mathbb{R}$. Now we may write

$$\mathbf{y} \cdot \mathbf{r} = c_1\beta_1 + \cdots + c_m\beta_m. \tag{2.7}$$

Recall that the c_i terms can individually vary over $\mathbb{R}$, and any combination of c_i terms is valid as long as not all of them are zero. With this in mind, let $n-1$ of the constants c_i be a real number such that not all of them are zero, and call the remaining constant c_p, where $1 \leq p \leq m$. We will show that (2.7) can always be zero with only a proper choice of c_p by analyzing two cases: when $\beta_p = 0$, and when $\beta_p \neq 0$.

(1) If $\beta_p = 0$, then $\beta_p = \mathbf{y} \cdot \mathbf{v}_p = 0$. Hence, the inner product $\mathbf{y} \cdot \mathbf{r} = 0$ for $\mathbf{r} = \mathbf{v}_p \in \Lambda \backslash \{\mathbf{0}\}$, and we are done.

(2) If $\beta_p \neq 0$, then define

$$c_p = -\frac{1}{\beta_p} \sum_{\substack{i=1 \\ i \neq p}}^{m} c_i\beta_i \in \mathbb{R}. \tag{2.8}$$

If we substitute this c_p in (2.7), the result is $\mathbf{y} \cdot \mathbf{r} = 0$. Hence if $\mathbf{r}$ is the vector constructed with the chosen constants c_i, including the c_p defined in (2.8), then $\mathbf{y} \cdot \mathbf{r} = 0$.

Since $\mathbf{y} \cdot \mathbf{r} = 0$ for both cases, the lemma is proved. □

The following theorem that we obtain from Lemma 2.3.1 is a part of linear systems theory used in [2, 3, 14], yet it has not been written explicitly in the form it is written in here (as far as we have seen in the literature). This is the first main theorem for our study, as it categorizes most completely uncontrollable graphs.

Theorem 2.3.1 (Repeated Eigenvalues)**.** *If the Laplacian matrix* $\boldsymbol{L} = \boldsymbol{L}(X)$ *of a graph* X *on* n *vertices has any repeated eigenvalues, then the LLFD of* X *are completely uncontrollable.*

Proof. Let Λ be an eigenspace of $\mathbf{L}$ with $\dim \Lambda \geq 2$ (equivalently, $\mathbf{L}$ has repeated eigenvalues). By Lemma 2.3.1, for every $\mathbf{b} \in \mathbb{R}^n$ there exists some non-zero $\mathbf{v} \in \Lambda$ such that $\mathbf{v}^{\mathrm{T}}\mathbf{b} = 0$. Then, by the PBH test (Lemma 1.4.2), the pair $(\mathbf{L}, \mathbf{b})$ is uncontrollable for all $\mathbf{b} \in \{0,1\}^n \backslash \{\mathbf{0}, \mathbf{1}\}$. Therefore, the LLFD of X are completely uncontrollable. $\square$

The following corollary is the contrapositive to Theorem 2.3.1.

Corollary 2.3.1. *If for some graph X there exists $\boldsymbol{b} \in \{0,1\}^n \backslash \{\boldsymbol{0}, \boldsymbol{1}\}$ such that the pair $(\boldsymbol{L}, \boldsymbol{b})$ is controllable, then the eigenvalues of the Laplacian matrix of X are all distinct.*

These results are important to the study, as our simulations suggest that the class of completely uncontrollable graphs is almost entirely comprised of graphs whose Laplacian matrices have repeated eigenvalues (Table 2.1). After discerning this argument, one might be lead to wonder if *all* completely uncontrollable graphs can be identified by having repeated eigenvalues, i.e. completely uncontrollable $\iff$ repeated eigenvalues. This prospect is interesting and non-intuitive, yet false. Indeed, there are examples of completely uncontrollable graphs which have distinct eigenvalues. By the PBH test (Lemma 1.4.2), we understand that such a graph must have an eigenvector $\mathbf{v}$ such that $\mathbf{v}^T\mathbf{b} = 0$ for all control vectors $\mathbf{b} \in \{0,1\}^n \backslash \{\mathbf{0}, \mathbf{1}\}$. Over all of the simulations ran through this study, only one graph was ever found which was completely uncontrollable and had distinct eigenvalues. The Laplacian matrix for this graph was unfortunately lost during the run time of the simulation. The study done by Augilar and Gharesifard [3] found ten graphs on eight nodes, and twelve graphs on nine nodes that had distinct eigenvalues and were also completely uncontrollable, Figure 2.1 depicts one on eight vertices. They agree that the class of completely uncontrollable graphs with distinct eigenvalues are a special class, and have potential for further research.

n	CU	RE
6	0.4028	0.4028
7	0.2136	0.2136
8	0.1646	0.1646
9	0.0792	0.0792
10	0.0490	0.0490
11	0.0180	0.0180
12	0.0148	0.0148
13	0.0046	0.0046
14	0.0018	0.0018
15	0.0010	0.0010

Table 2.1: Proportion of completely uncontrollable vs. repeated eigenvalues graphs.

For each vertex set size n, our simulation generated ten thousand Laplacian matrices of Erdős-Rényi random graphs with edge density $p = 1/2$, not up to isomorphism, and including repetitions (see Appendix B). The program tested each graph to find its controllability class. For those graphs that were completely uncontrollable (CU), the program also checked to see if the matrix had repeated eigenvalues (RE). As the table shows, out of the total one hundred thousand random matrices generated, every matrix that was completely uncontrollable also had repeated eigenvalues. This is partially because the proportion of completely uncontrollable graphs decreases with increasing n, meaning it becomes difficult to find completely uncontrollable graphs at large enough n. Hence the sub-class of completely uncontrollable graphs with repeated eigenvalues appears to be a small fraction of an already small portion of graphs.

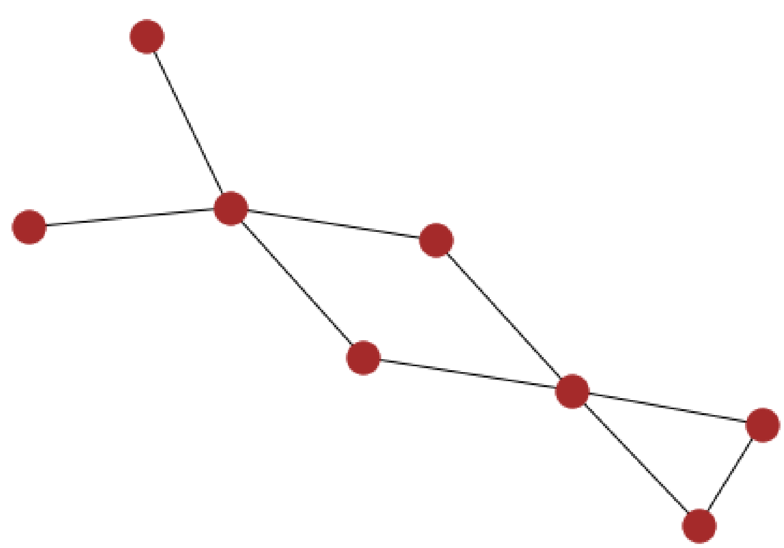

Figure 2.1: A completely uncontrollable graph with distinct eigenvalues.

This graph was found by C. Augilar and B. Gharesifard in [3].

2.4 Disconnected Graphs

The first result we can formulate with Theorem 2.3.1 is the uncontrollability of disconnected graphs. Before this, we will prove a lemma regarding **block matrices**, which are matrices whose elements are also matrices.

Lemma 2.4.1. *Let $\boldsymbol{A}_1, \dots, \boldsymbol{A}_m$ be a collection of m square matrices indexed by $1 \leq i \leq m$, with $m \geq 2$. We use n_i to denote the dimension of matrix $\boldsymbol{A}_i$. If $\boldsymbol{A}_j$ has $\boldsymbol{v}$ as an eigenvector with eigenvalue λ, then the diagonal block matrix*

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{A}_1 & & 0 \\ & \ddots & \\ 0 & & \boldsymbol{A}_m \end{pmatrix}, \tag{2.9}$$

has eigenvalue λ with associated eigenvector $\boldsymbol{v'} = [\boldsymbol{q}_1, \dots, \boldsymbol{q}_m]^T$, where $\boldsymbol{q}_i = \begin{cases} \boldsymbol{0}_{n_i} \text{ if } i \neq j \\ \boldsymbol{v} \text{ if } i = j \end{cases}$.

Proof.

$$\mathbf{A}\mathbf{v}' = \begin{pmatrix} \mathbf{A}_1 & & 0 \\ & \ddots & \\ 0 & & \mathbf{A}_m \end{pmatrix} \begin{pmatrix} \mathbf{q}_1 \\ \vdots \\ \mathbf{q}_m \end{pmatrix} = [\mathbf{A}_1\mathbf{q}_1, \dots, \mathbf{A}_m\mathbf{q}_m]^{\mathrm{T}} = \lambda\,[\mathbf{q}_1, \dots, \mathbf{q}_m]^{\mathrm{T}} = \lambda\mathbf{v}'. \tag{2.10}$$

□

We will relate this lemma to our study of the Laplacian matrix, but first we need to define the concept of connectedness for graphs, following Godsil [6].

Definition 2.4.1 (Disconnected Graph)**.** *We define a **path** in a graph as a sequence of vertices such that consecutive vertices are neighbors. If there are two vertices in a graph which have no path between them, we say the graph is **disconnected**. If every pair of vertices has at least one path between them, then the graph is **connected**.*

Figure 2.2 depicts a disconnected graph. Now we can relate the idea in Lemma 2.4.1 to graph connectedness.

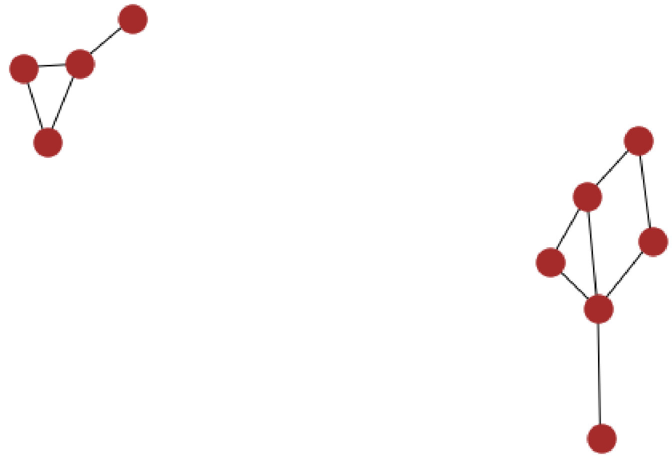

Figure 2.2: A disconnected graph, composed of two independent connected graphs.

The subgraph composed of the vertices on the left has conditionally controllable LLFD, and the subgraph composed of the vertices on the right has essentially controllable LLFD. However, Theorem 2.4.1 tells us the overall graph has completely uncontrollable LLFD.

Theorem 2.4.1. *If a graph X is disconnected, then the LLFD of X are completely uncontrollable.*

Proof. Since X is disconnected, let $X_1, \ldots, X_m$ $(m \geq 2)$ be the set of all disjoint subgraphs of X such that for any vertex $x_i \in X_i$, the graph X_i is the largest connected subgraph of X containing vertex x_i for $1 \leq i \leq m$. We will now construct the graph $Y \cong X$, where Y has disjoint subgraphs $Y_1, \ldots, Y_m$ such that $Y_i \cong X_i$. Denote the number of vertices in the subgraph $X_i \cong Y_i$ with n_i. Construct Y such that

$$V(Y_i) = \begin{cases} \{1, \ldots, n_i\} & i = 1 \\ \\ \{n_{i-1}+1, \ldots, n_{i-1}+n_i\} & i > 1 \end{cases} \tag{2.11}$$

(this graphs exists by our discussion in Chapter 1). With this condition on the vertex sets of each Y_i, the Laplacian matrix of Y is

$$\mathbf{L}(Y) = \begin{pmatrix} \mathbf{L}(Y_1) & & 0 \\ & \ddots & \\ 0 & & \mathbf{L}(Y_m) \end{pmatrix}. \tag{2.12}$$

Notice that (2.12) has the same form as (2.9). Each $\mathbf{L}(Y_i)$ in (2.12) is a Laplacian matrix of some subgraph of Y, and thus has eigenvector $\mathbf{1}_{n_i}$ with eigenvalue 0 as shown in Theorem 2.2.1. Using

Lemma 2.4.1, the matrix $\mathbf{L}(Y)$ has two eigenvectors of the form $[\mathbf{q}_1, \ldots, \mathbf{q}_m]^{\mathrm{T}}$: the first where $\mathbf{q}_1 = \mathbf{1}_{n_1}$ and every other $\mathbf{q}_i = \mathbf{0}_{n_i}$, and the second where $\mathbf{q}_m = \mathbf{1}_{n_m}$ and every other $\mathbf{q}_i = \mathbf{0}_{n_i}$. These two eigenvectors are clearly linearly independent, and both have eigenvalue zero. Hence we can conclude that $\mathbf{L}(Y)$ has eigenvalue 0 with multiplicity ≥ 2, and the LLFD system with $\mathbf{L}(Y)$ is completely uncontrollable by Theorem 2.3.1. Since $X \cong Y$, the same is true for the LLFD system with $\mathbf{L}(X)$ by Lemma 1.6.1. $\square$

Because of Theorem 2.4.1, we will only consider connected graphs for the remainder of the paper.

Chapter 3

Circulant Graphs

This chapter focuses on circulant graphs, which are also studied in [11, 14]. Circulant graphs can be characterized by having cyclic nature, which we will show causes the Laplacian matrices of circulant graphs to have repeated eigenvalues. After explicitly defining circulant graphs and matrices, we will introduce and utilize some characteristics of circulant matrices which will ultimately show that circulant graphs are completely uncontrollable.

3.1 Definitions

Definition 3.1.1 (circulant graph). *Let* $\mathbb{Z}_n$ *be the group of integers under addition modulo* n. *Let* $C \subseteq \mathbb{Z}_n \backslash 0$ *be a subset , with the condition that* C *is closed with respect to inverses, i.e.* $c \in C \iff -c \in C$. *Now construct the graph* X *with* n *vertices such that the pair* $\{i, j\}$ *is an edge of* X *if and only if* $(i - j)$ *mod* $n \in C$. *The graph* X *is called* ***circulant*** *with connection set* C.

Figure 3.1 depicts a circulant graph on eight vertices with connection set $C = \{-2, -1, 1, 2\}$. Notice how the graph has two cyclic subgraphs: $\{0, 2, 4, 6\}$ and $\{1, 3, 5, 7\}$. Next, we define a circulant matrix (independent from the notion of circulant graphs).

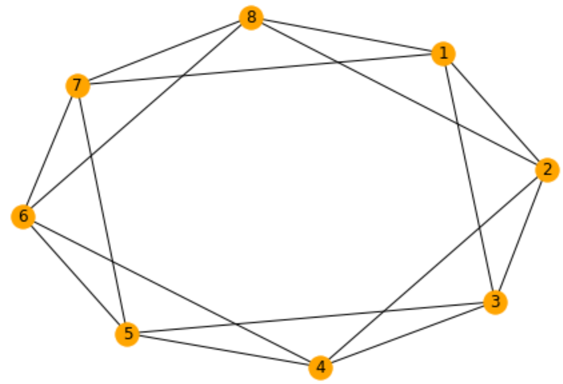

Figure 3.1: The circulant graph on eight nodes with $C = \{-2, -1, 1, 2\}$.

Definition 3.1.2 (circulant matrix)**.** *For some $a_1, a_2, \ldots, a_n \in \mathbb{R}$, we define the* ***circulant matrix*** $\mathrm{circ}(a_1, a_2, \ldots, a_n)$ *as*

$$\mathrm{circ}(a_1, a_2, \ldots, a_n) = \begin{pmatrix} a_1 & a_n & a_{n-1} & \ldots & a_2 \\ a_2 & a_1 & a_n & \ldots & a_3 \\ a_3 & a_2 & a_1 & \ldots & a_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n-1} & a_{n-2} & \ldots & a_1, \end{pmatrix}. \tag{3.1}$$

Observe that every column is a cyclic iteration of the first column, and that every row is a cyclic iteration of the first row. In the next section, we use Toeplitz matrix theory from Chapter 2 of [4] to find an explicit expression for the eigenvalues of a circulant matrix.

3.2 Circulant Matrices

Circulant matrices are a subclass of the more general Toeplitz matrices, which is the type of matrix studied in [4]. Chapter 2 in that text is specifically reserved for circulant matrices, which have a connection to Fourier transforms. The following lemma is mentioned in this text.

Lemma 3.2.1 (Eigenvalues of Circulant Matrices)**.** *For some $a_1, a_2, \ldots, a_n \in \mathbb{R}$, define $a(z) = a_1 + a_2 z + \cdots + a_n z^{n-1}$ and $\omega_n = e^{2\pi i/n}$. Then the eigenvalues of $\mathrm{circ}(a_1, a_2, \ldots, a_n)$, denoted λ_m*

for index $0 \leq m \leq n-1$*, including multiplicity, are*

$$\lambda_0 = a(\omega_n^0),\ \lambda_1 = a(\omega_n^1),\ \ldots,\ \lambda_{n-1} = a(\omega_n^{n-1}). \tag{3.2}$$

This fact is proved in [18]. Next, we analyze a particular class of circulant matrices.

Lemma 3.2.2 (Eigenvalues of Real Symmetric Circulant Matrices)**.** *The eigenvalues of real symmetric circulant matrices of dimension* n *are given by*

$$\lambda_m = \sum_{j=1}^{n} a_j \cos[2\pi m(j-1)/n],\ 0 \leq m \leq n-1, \tag{3.3}$$

including multiplicity.

Proof. Let $\mathbf{A}$ be a real symmetric circulant matrix. $\mathbf{A}$ takes the form

$$\mathbf{A} = \begin{pmatrix} a_1 & a_2 & a_3 & \ldots & a_n \\ a_2 & a_1 & a_2 & \ldots & a_{n-1} \\ a_3 & a_2 & a_1 & \ldots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n-1} & a_{n-2} & \ldots & a_1 \end{pmatrix}. \tag{3.4}$$

From Lemma 3.2.1, we know that the eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ of this matrix are

$$\lambda_0 = a(\omega_n^0),\ \lambda_1 = a(\omega_n^1),\ \ldots,\ \lambda_{n-1} = a(w_n^{n-1}), \tag{3.5}$$

where $a(z) = \sum_{j=1}^{n} a_j z^{j-1}$ and $\omega_n = e^{2\pi i/n}$. Then the m-th eigenvalue is

$$\lambda_m = a(\omega_n^m) = \sum_{j=1}^{n} a_j (\omega_n^m)^{j-1} = \sum_{j=1}^{n} a_j \exp\left[2\pi i m(j-1)/n\right],\quad 0 \leq m \leq n-1. \tag{3.6}$$

Now we will deconstruct (3.6) into sines and cosines:

$$\begin{aligned} \lambda_m &= \sum_{j=1}^{n} a_j \exp\left[2\pi i m(j-1)/n\right] \\ &= \sum_{j=1}^{n} a_j \left[\cos\left(\frac{2\pi m(j-1)}{n}\right) + i \sin\left(\frac{2\pi m(j-1)}{n}\right)\right]. \end{aligned} \tag{3.7}$$

$\mathbf{A}$ is real symmetric, so the eigenvalues of $\mathbf{A}$ are real by Lemma 1.7.6. Therefore we know the imaginary part of (3.7) will end up being zero, which gives

$$\lambda_m = \sum_{j=1}^{n} a_j \cos\left(\frac{2\pi m(j-1)}{n}\right). \tag{3.8}$$

□

Now equipped with Lemma 3.2.2, we can show that the eigenvalues of any real symmetric circulant matrix (for $n \geq 3$ vertices) are not all distinct.

Lemma 3.2.3. *Any real symmetric circulant matrix of dimension $n \geq 3$ has repeated eigenvalues.*

Proof. Let $\mathbf{A}$ be a real symmetric circulant matrix so that $\mathbf{A} = \text{circ}(a_1, \ldots, a_n)$ for $a_1, \ldots, a_n \in \mathbb{R}$. We know from the previous lemma that

$$\lambda_m = \sum_{j=1}^{n} a_j \cos\left(\frac{2\pi m(j-1)}{n}\right), \ 0 \leq m \leq n-1. \tag{3.9}$$

Let $m_1 \in \mathbb{Z}$ such that $0 < m_1 < \frac{n}{2}$. Then

$$\lambda_{m_1} = \sum_{j=1}^{n} a_j \cos\left(\frac{2\pi m_1(j-1)}{n}\right). \tag{3.10}$$

Now let $m_2 = n - m_1$, acknowledging that $m_2 \in \mathbb{Z}$ such that $\frac{n}{2} < m_2 < n$ by our definition of m_1. Then

$$\lambda_{m_2} = \sum_{j=1}^{n} a_j \cos\left(\frac{2\pi m_2(j-1)}{n}\right) = \sum_{j=1}^{n} a_j \cos\left(\frac{2\pi(n-m_1)(j-1)}{n}\right). \tag{3.11}$$

We know that the cosine function is cyclic with a period of integer multiples of 2π. Explicitly:

$$\cos(2\pi k + z) = \cos(z) \tag{3.12}$$

for integer k, see [1]. Using this fact, we have

$$\cos\left(\frac{2\pi k(n-x)}{n}\right) = \cos\left(2\pi k\left(1 - \frac{x}{n}\right)\right) = \cos\left(-\frac{2\pi kx}{n}\right) = \cos\left(\frac{2\pi kx}{n}\right). \tag{3.13}$$

Then

$$\lambda_{m_2} = \sum_{j=1}^{n} a_j \cos\left(\frac{2\pi(n-m_1)(j-1)}{n}\right) = \sum_{j=1}^{n} a_j \cos\left(\frac{2\pi(j-1)m_1}{n}\right) = \lambda_{m_1} \tag{3.14}$$

by (3.13). Since $m_1 \neq m_2$ by definition, we can conclude that λ_1 and λ_2 are a pair of repeated eigenvalues of $\mathbf{A}$. $\square$

This connection between discreet Fourier transforms and circulant matrices that is highlighted in [4] will be vital to studying the Laplacian matrix of circulant graphs.

3.3 The Uncontrollability of Circulant Graphs

If the Laplacian matrix of any circulant graph on $n \geq 3$ vertices is a real symmetric circulant matrix, then Lemma 3.2.3 from the previous section guarantees that the Laplacian matrix will have repeated eigenvalues. Applying the repeated eigenvalue theorem (Theorem 2.3.1), we can conclude that circulant graphs are completely uncontrollable. This section aims to repeat that argument rigorously by showing that the Laplacian matrix will have the same form as circulant matrices.

Lemma 3.3.1. *Circulant graphs are regular.*

Proof. By definition, each vertex i in a circulant graph X is connected to a vertex j if and only if $(i-j) \bmod n \in C$, where C the connection set of X. For each vertex i, there are $|C|$ different j such that $(i-j) \bmod n \in C$, so i has $|C|$ neighbors. This applies to all vertices in X, so we conclude that X is $|C|$-regular. $\square$

Lemma 3.3.2. *The Laplacian matrix of a circulant graph is a real symmetric circulant matrix.*

Proof. Let X be a circulant graph on n nodes with connection set C, and define $k = |C|$ so that X is k-regular (Lemma 3.3.1). Then the Laplacian matrix has the form

$$\mathbf{L} = \begin{pmatrix} k & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & k & a_{22} & \dots & a_{2n} \\ a_{31} & a_{32} & k & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & k \end{pmatrix}, \qquad a_{ij} = \begin{cases} -1 & \text{if } (i-j) \bmod n \in C \\ 0 & \text{if } (i-j) \bmod n \notin C \end{cases}. \tag{3.15}$$

Recall from graph theory that the Laplacian matrix is symmetric. Hence we have

$$\mathbf{L} = \begin{pmatrix} k & a_{12} & a_{13} & \dots & a_{1n} \\ a_{12} & k & a_{22} & \dots & a_{2n} \\ a_{13} & a_{23} & k & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & a_{3n} & \dots & k \end{pmatrix}, \qquad a_{ij} = \begin{cases} -1 & \text{if } (i-j) \bmod n \in C \\ 0 & \text{if } (i-j) \bmod n \notin C \end{cases}. \tag{3.16}$$

Notice that if $(i-j) \bmod n \in C$, then $[(i+1)-(j+1)] \bmod n = (i-j) \bmod n \in C$, and if $(i-j) \bmod n \notin C$, then $[(i+1)-(j+1)] \bmod n = (i-j) \bmod \ \notin C$. Following this, we have $a_{ij} = -1 \implies a_{i+1\ j+1} = -1$ and $a_{ij} = 0 \implies a_{i+1\ j+1} = 0$. This means that each row is a cyclic iteration of the first row. The j-th entry of the i-th row is the same as the $(j-1)$-th entry in the $(i-1)$-th row. With this, we can write $\mathbf{L}$ as

$$\mathbf{L} = \begin{pmatrix} k & a_{12} & a_{13} & \dots & a_{1n} \\ a_{1n} & k & a_{12} & \dots & a_{1\ n-1} \\ a_{1\ n-1} & a_{1n} & k & \dots & a_{1\ n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{12} & a_{13} & a_{14} & \dots & k \end{pmatrix}, \tag{3.17}$$

which is clearly a circulant matrix. Since $\mathbf{L}$ takes the form of both (3.16) and (3.17), we can conclude that $\mathbf{L}$ is a real symmetric circulant matrix. $\square$

Now, as stated in the beginning of this section, we will use Lemma 3.2.3 and Lemma 3.3.2 to make a general statement about the controllability of circulant graphs.

Theorem 3.3.1. *All circulant graphs on $n \geq 3$ vertices are completely uncontrollable.*

Proof. This theorem comes as a corollary of Lemma 3.2.3 and Lemma 3.3.2. If some circulant graph has a Laplacian matrix $\mathbf{L}$, then Lemma 3.3.2 tells us that $\mathbf{L}$ is real symmetric. By Lemma 3.2.3, $\mathbf{L}$ has repeated eigenvalues. Since $\mathbf{L}$ has repeated eigenvalues, Lemma 2.3.1 guarantees that the LLFD of any circulant graph are completely uncontrollable. $\square$

Chapter 4

Bipartite Graphs

In this chapter, we perform an analysis of bipartite graphs, similar to the previous chapter for circulant graphs. Unlike circulant graphs, however, bipartite graphs are not in general completely uncontrollable. Instead, we prove that there is a bound on the relative size of certain subsets of the vertices in bipartite graphs. First, we need to explain exactly what characteristics of bipartite graphs are necessary for the analysis.

4.1 Definitions

Definition 4.1.1 (Bipartite and Biregular Graphs)**.** *If the vertex set of a graph X can be partitioned into two disjoint sets U and V such that for all vertices $u_1, u_2 \in U$ and $v_1, v_2 \in V$, we have that $\{u_1, u_2\}$ and $\{v_1, v_2\}$ are not edges of X, we say X is* ***bipartite****. Furthermore, if every vertex in the set U has the same number of neighbors and every vertex in the set V has the same number of neighbors, we say X is* ***biregular****.*

Given a graph X, the sets U and V from Definition 4.1.1 are called the **parts** of X. If $|U| \geq |V|$ and every vertex in U has the same number of neighbors, then we say X is **semi-regular**. Figure 4.1 depicts a bipartite graph that is semi-regular, and Figure 4.2 depicts a bipartite graph that is also biregular.

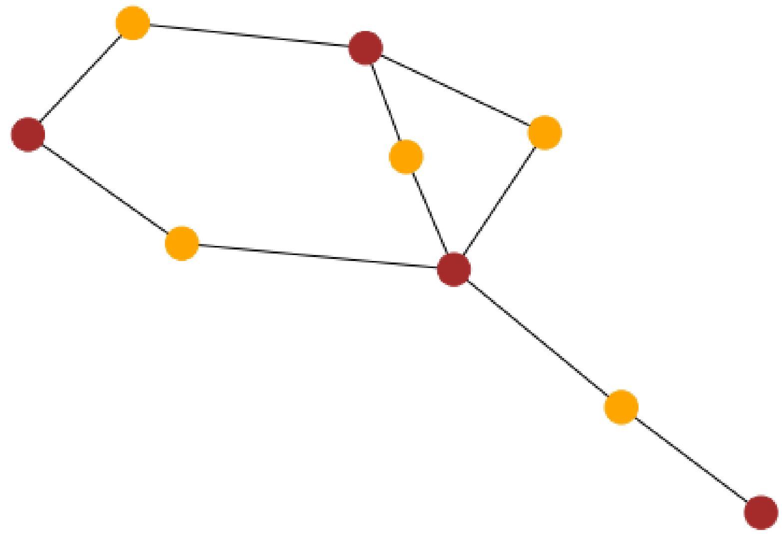

Figure 4.1: A bipartite graph with nine vertices.

Observe that each orange node is connected to two brown nodes. Since there are more brown nodes than orange nodes, and each brown nodes has two neighbors, this graph is semi-regular. However, there is no common number of connections for the brown nodes, so this graph is not biregular. The LLFD system with the Laplacian of this graph is conditionally controllable.

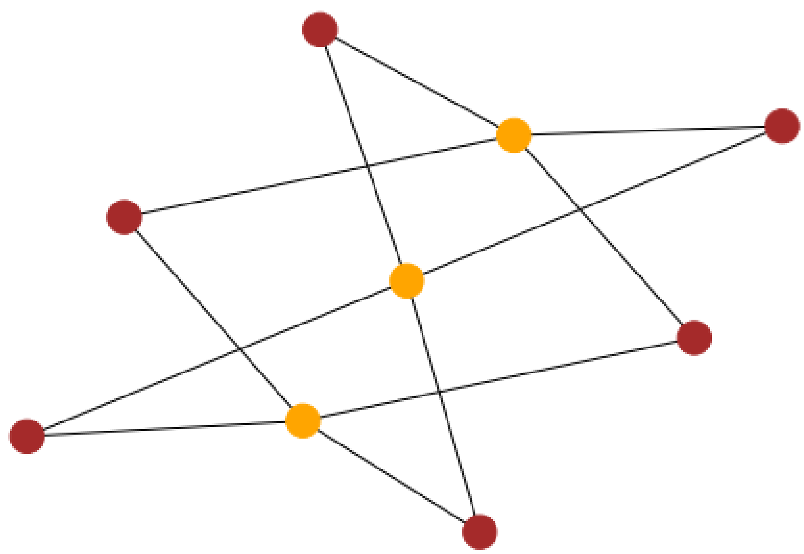

Figure 4.2: A biregular graph with nine vertices.

Notice that the brown vertices all have two neighbors and the orange vertices all have four neighbors, so Figure 4.2 is a biregular graph (it is also semi-regular). The LLFD system with the Laplacian of this graph is completely uncontrollable.

4.2 The Uncontrollability of Bipartite Graphs

Given a bipartite graph X whose partition has the parts U and V such that $u = |U| \geq |V| = v$, this section will show that there exists some u_{min} such that if $u \geq u_{min}$, then the LLFD of X are completely uncontrollable.

Lemma 4.2.1. *Let X be a connected bipartite graph on $n \geq 3$ vertices whose partition has the parts U and V such that $u = |U| \geq |V| = v$. Let $1 \leq k \leq v$ represent one of the possible degrees of the nodes in U. If at least $\binom{v}{k} + 2$ of the vertices in U have k neighbors, then k is a repeated eigenvalue of the Laplacian matrix of X.*

Proof. Let $\mathbf{L} = \mathbf{L}(X)$ be the Laplacian matrix of X, and let $I \subseteq U$ be the set of indices of vertices in U which have degree k. Let $\mathbf{x}_i$ be the i-th column of $\mathbf{L}$. For any vertex $i \in I$, there are $\binom{v}{k}$ possible choices of edge sets, since i can only have edges with vertices in V. The column $\mathbf{x}_i$ has k in the i-th entry, k entries of -1 (which correspond to the edges of vertex i), and zero elsewhere. Now consider the shifted Laplacian $\mathbf{L} - k\mathbf{I}$, where $\mathbf{y}_i$ is the i-th column of $\mathbf{L} - k\mathbf{I}$. We have that $\mathbf{y}_i = \mathbf{x}_i - k\mathbf{e}_i$, so the column $\mathbf{y}_i$ has k entries of -1 (which again correspond to the edges of vertex i), and zero elsewhere. Since $\mathbf{y}_i$ has k non-zero entries and only v places to put them, there are exactly $\binom{v}{k}$ possible arrangements that $\mathbf{y}_i$ can take. Now suppose that $|I| \geq \binom{v}{k} + 2$. By the pigeonhole principle, there exists distinct vertices $a, b, c, d \in I$ such that one of the following must be true:

(1) there are at least two pairs of equivalent columns: $\mathbf{y}_a = \mathbf{y}_b \neq \mathbf{y}_c = \mathbf{y}_d$,

(2) there is a collection of at least three columns that are equal: $\mathbf{y}_a = \mathbf{y}_b = \mathbf{y}_c$.

We show that both situations give rise to repeated eigenvalues:

(1) If $\mathbf{y}_a = \mathbf{y}_b \neq \mathbf{y}_c = \mathbf{y}_d$ are columns of $\mathbf{L} - k\mathbf{I}$, then $\mathbf{e}_a - \mathbf{e}_b$ and $\mathbf{e}_c - \mathbf{e}_d$ are linearly independent eigenvectors of $\mathbf{L} - k\mathbf{I}$ with eigenvalue zero.

(2) If $\mathbf{y}_a = \mathbf{y}_b = \mathbf{y}_c$ are columns of $\mathbf{L} - k\mathbf{I}$, then $\mathbf{e}_a - \mathbf{e}_b$ and $\mathbf{e}_a - \mathbf{e}_c$ are linearly independent eigenvectors of $\mathbf{L} - k\mathbf{I}$ with eigenvalue zero.

Hence zero is a repeated eigenvalue of $\mathbf{L} - k\mathbf{I}$. By Lemma 1.7.7, this means that k is a repeated eigenvalue of $\mathbf{L}$. $\square$

We will now apply this idea to every vertex degree at the same time.

Theorem 4.2.1. *Let X be a connected bipartite graph on $n \geq 3$ vertices whose partition has the parts U and V such that $u = |U| \geq |V| = v$. If*

$$u \geq 1 + v + \sum_{j=1}^{v} \binom{v}{j}, \tag{4.1}$$

then the LLFD of X are completely uncontrollable.

Proof. For each vertex in U, there are v choices of degree. Let k be one of the degree choices $1 \leq k \leq v$ for vertices in U. Invoking the pigeonhole principle again, if

$$u \geq 1 + \left[\binom{v}{1} + 1\right] + \left[\binom{v}{2} + 1\right] + \cdots + \left[\binom{v}{v} + 1\right] = 1 + \sum_{j=1}^{v} \left[\binom{v}{j} + 1\right], \tag{4.2}$$

then for at least one degree choice k there are at least $\binom{v}{k} + 2$ vertices with degree k. With this knowledge, Lemma 4.2.1 guarantees that k is a repeated eigenvalue of $\mathbf{L}$. Therefore, if

$$u \geq 1 + v + \sum_{j=1}^{v} \binom{v}{j}, \tag{4.3}$$

then the LLFD of X are completely uncontrollable by Theorem 2.3.1. $\square$

Theorem 4.2.1 is the main result for this section, but we can discuss the form of the result further. Using the OEIS, we noticed that

$$\sum_{k=1}^{n} \binom{n}{k} = 2^n - 1 \tag{4.4}$$

for positive integers n. To understand why this is true, consider the number of committees of people one can form from a selection of n people. From one perspective, we can make a committee of one person, two people, ..., up to n people, and there are $\binom{n}{k}$ possible committees for each committee size k, i.e. $\sum_{k=1}^{n} \binom{n}{k}$. From another perspective, we can have all of the people line up, and we can one-by-one assign each of them to be either in the committee or not in the committee, which is

2^n possibilities. We do not consider the case of choosing nobody to be in the committee, so we subtract one from 2^n. Hence $\sum_{k=1}^{n}\binom{n}{k}$ and $2^n - 1$ are both the number of possible committees to form from n people. Thus we can restate Theorem 4.2.1 in a slightly more intuitive form as a corollary:

Corollary 4.2.1. *Let X be a connected bipartite graph on $n \geq 3$ vertices whose partition has the parts U and V such that $u = |U| \geq |V| = v$. If*

$$u \geq 2^v + v \tag{4.5}$$

then the LLFD of X are completely uncontrollable.

Proof. From the previous discussion prior to the corollary and Theorem 4.2.1,

$$u \geq 1 + v + \sum_{j=1}^{v}\binom{v}{j} = 1 + v + 2^v - 1 = 2^v + v. \tag{4.6}$$

□

The expression (4.5) highlights that this bound becomes very large very fast, which means that as the smaller vertex set V gets larger, the minimum size of U which forces the LLFD of X to be completely uncontrollable grows very quickly.

4.3 Regularity and Uncontrollability

Contrary to the results of the previous section, bipartite graphs which are also semi-regular have a much more strict bound on the minimum size for uncontrollability. In this section, we show that if X is a semi-regular bipartite graph whose partition has the parts U and V, where the LLFD of X are essentially controllable or conditionally controllable, then either $|U| = |V|$ or $|U| = |V| + 1$. To prove this, we will show that if $|U| > |V| + 1$, then the Laplacian matrix of X has repeated eigenvalues.

Lemma 4.3.1. *Let X be a connected bipartite graph on $n \geq 3$ vertices whose partition has the parts U and V such that $u = |U| \geq |V| = v$, so that $n = u + v$. Suppose that every vertex in U has k neighbors. If $u > v + 1$, then k is an eigenvalue of the Laplacian of X, which has multiplicity > 1.*

Proof. As we have defined the graph X above, we can write its Laplacian as

$$\mathbf{L} = \begin{pmatrix} k\mathbf{I}_u & \mathbf{Q}^T \\ \mathbf{Q} & \mathbf{D}_V \end{pmatrix}, \tag{4.7}$$

where $\mathbf{I}_u$ is the $u \times u$-dimensional identity matrix, $\mathbf{D}_V$ is the $v \times v$-diagonal matrix whose entries are the degrees of vertices $v \in V$, and $\mathbf{Q}$ is a $v \times u$-dimensional matrix with either 0 or -1 as entries, depending on the edges in E. Now consider the following shift on the diagonal of $\mathbf{L}$:

$$\mathbf{L} - k\mathbf{I} = \begin{pmatrix} \mathbf{0}_u & \mathbf{Q}^T \\ \mathbf{Q} & \mathbf{D}_V - k\mathbf{I}_v \end{pmatrix}, \tag{4.8}$$

where $\mathbf{0}_u$ is the $u \times u$-dimensional square matrix with all zero entries, $\mathbf{I}_v$ is the $v \times v$-dimensional identity matrix, and $\mathbf{I}$ is the $n \times n$-dimensional identity matrix. Define

$$\mathbf{S} = \begin{pmatrix} \mathbf{0}_u \\ \mathbf{Q} \end{pmatrix} \text{ and } \mathbf{T} = \begin{pmatrix} \mathbf{Q}^T \\ \mathbf{D}_V - k\mathbf{I} \end{pmatrix} \tag{4.9}$$

so that

$$\mathbf{L} - k\mathbf{I} = \begin{pmatrix} \mathbf{S} & \mathbf{0}_{n\times v} \end{pmatrix} + \begin{pmatrix} \mathbf{0}_{n\times u} & \mathbf{T} \end{pmatrix}, \tag{4.10}$$

where $\mathbf{S}$ is $n\times u$-dimensional and $\mathbf{T}$ is $n\times v$-dimensional. By Lemma 1.7.2, we have $\operatorname{rank}\mathbf{S} = \operatorname{rank}\mathbf{S}^T$, so

$$\operatorname{rank}\mathbf{S} = \operatorname{rank}\mathbf{S}^T = \operatorname{rank}\begin{pmatrix} \mathbf{0}_u & \mathbf{Q}^T \end{pmatrix} = \dim\left(\operatorname{col}\left(\mathbf{Q}^T\right)\right) \leq v, \tag{4.11}$$

since $\mathbf{Q}^T$ has v columns. Now, using Lemma 1.7.4, we have

$$\operatorname{rank}(\mathbf{L} - k\mathbf{L}) = \operatorname{rank}\left[\begin{pmatrix} \mathbf{S} & \mathbf{0} \end{pmatrix} + \begin{pmatrix} \mathbf{0} & \mathbf{T} \end{pmatrix}\right] \leq \operatorname{rank}\mathbf{S} + \operatorname{rank}\mathbf{T} \leq v + v = 2v. \tag{4.12}$$

In other words, the maximum rank $\mathbf{L} - k\mathbf{I}$ can obtain is $2v$. Now by the rank-nullity theorem (Lemma 1.7.3), we have

$$\operatorname{rank}(\mathbf{L} - k\mathbf{I}) + \dim\left[\ker(\mathbf{L} - k\mathbf{I})\right] = n \text{ or } \dim\left[\ker(\mathbf{L} - k\mathbf{I})\right] = n - \operatorname{rank}(\mathbf{L} - k\mathbf{I}). \tag{4.13}$$

By what we have just shown, the dimension of the null space of $\mathbf{L} - k\mathbf{I}$ is bounded as

$$\dim\left[\ker(\mathbf{L} - k\mathbf{I})\right] \geq n - 2v = u + v - 2v = u - v. \tag{4.14}$$

We can now see that if $u > v+1 \implies u-v > 1$, then $\dim[\ker(\mathbf{L} - k\mathbf{I})] > 1$, which means that zero is repeated eigenvalue of $\mathbf{L} - k\mathbf{I}$ by Lemma 1.7.5. Since $\mathbf{L} - k\mathbf{I}$ has eigenvalue 0 with multiplicity > 1, Lemma 1.7.7 says then $\mathbf{L}$ has eigenvalue k with multiplicity > 1. □

Now we can formulate a statement regarding the controllability of semi-regular bipartite graphs. The next theorem follows directly from the lemma we just proved.

Theorem 4.3.1. *Let X be a connected bipartite graph on $n \geq 3$ vertices whose partition has the parts U and V such that $u = |U| \geq |V| = v$. Suppose that every vertex in U has the same number of neighbors. If $u > v + 1$, then the LLFD of X are completely uncontrollable.*

Proof. Let k be the degree of vertices in U, and suppose $u > v + 1$. Lemma 4.3.1 says that k is a repeated eigenvalue of the Laplacian matrix of X. Finally, the repeated eigenvalue theorem (Theorem 2.3.1) gives us that the LLFD of X are completely uncontrollable. □

Biregular graphs are clearly semi-regular, so Theorem 4.3.1 applies. Considering the contrapositive of Theorem 4.2.1, we get the following corollary.

Corollary 4.3.1. *Let X be a connected bipartite graph on $n \geq 3$ vertices whose partition has the parts U and V such that $u = |U| \geq |V| = v$. If the LLFD of X are essentially controllable or conditionally controllable, then*

$$u = \begin{cases} v & \text{if } n \text{ is even,} \\ v+1 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. From Theorem 4.2.1, we know $u \leq v + 1 \implies u = v + 1$ or $u = v$. If n is even, then $u = v$ since $u = v + 1$ implies n is odd. If n is odd, then $u = v + 1$ since $u = v$ implies n is even. □

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Appendix A

Future Research

The initial stages of this project mostly involved developing the Python program to make simulations and realizing the concepts discussed in Chapter 2. After this point of progress, much of the research was choosing a popular class of graph (like circulant or bipartite), and generating many of these types of graphs, adding and changing certain specifications in order to find a pattern relating to controllability. There can be more time spent in this research, i.e. trying to find more classes of graphs whose structure guarantees repeated eigenvalues. A more interesting prospect is the mysterious nature of completely uncontrollable graphs with distinct eigenvalues, as mentioned in Chapter 2. This class is very rare, and it is unclear what property makes the LLFD uncontrollable, if not having repeated eigenvalues. Finally, this paper did not give much thought to the class of conditionally controllable graphs. The properties of these graph could also be related to what makes a graph with distinct eigenvalues have completely uncontrollable LLFD, as conditionally controllable graphs also must have distinct eigenvalues (Theorem 2.3.1).

Appendix B

Simulations

A substantial portion of the work done in this study is the Python 3 program given below which was used to study patterns in random matrices. The program generates the Laplacian matrix of the Erdős-Rényi random graph $G(n, 1/2)$ with n vertices and edge density $p = 1/2$, corresponding to a random sampling of graph over a uniform distribution of all graphs with n nodes. The script then finds the eigenvectors of the Laplacian matrix, and calculates the inner product of each eigenvector with each binary vector, storing every value. The PBH test (Theorem 1.4.2) then tells us exactly what class the graph belongs to, based off of the number of inner products that are zero and non-zero. Tables 1.1 and 2.1 were created using this program. It is important to note that, as the graph $G(n, 1/2)$ is a random graph, Tables 1.1 and 2.1 likely contain misleading results on two accounts. First, it is entirely possible that the simulation randomly generated and tested the Laplacian matrix of the exact same graph more than once, which would inflate the frequency of whichever controllability class contains the repeated graph. Similarly, it is also very possible that the simulation randomly generated and tested the Laplacian matrix of a series of isomorphic graphs, which would also inflate the frequency of the controllability class that the isomorphic graphs belong to. Despite this possibility, our data served to show us a pattern which we could eventually explain, and to that end the results of the simulations were essential to the study.

Random Matrix Generation

```
import numpy as np

# These first five functions are for generating random Laplacian matrices.

def gen_mat_element(i,j):
    """
    Generates a random element for entry (i,j) of an adjacency matrix.
    """
    if i==j:
        return 0
    if j<i:
        return 0
    else:
        return np.random.choice([0,1])

def sym_mat(matrix):
    """
    Symmetrizes an upper trinagle matrix with zero on the diagonal.

    """
    return matrix+np.transpose(matrix)

def gen_adj_mat(n):
    """
    Generates a random adjacency matrix.

    *n is the dimension of the matrix/number of vertices.

    """
    result = []

    for i in range(1,n+1):
        row = []
        for j in range(1,n+1):
            row.append(gen_mat_element(i,j))
        result.append(row)

    return sym_mat(result)


def sum_mat_rows(mat):
    """
    Returns an array whose entries are the
    sum of the rows of the input matrix "mat".

    """
    result = []

    for i in range(n):
        result.append(sum(mat[i]))

    return result
```

```
def gen_lap_mat(n):
    """
    Generates a random Laplacian matrix.

    *n is the dimension of the matrix/number of vertices.

    """
    adj_mat = gen_adj_mat(n)
    diag_mat = np.identity(n)*sum_mat_rows(adj_mat)

    return diag_mat - adj_mat

# This function will generate a random Laplacian matrix which
# describes a bipartite graph.

def gen_bipart_mat(a,b):
    """
    Generates a random bipartite graph.

    * a = number of vertices in one of the disjoint vertex sets.
    * b = number of vertices in other disjoint vertex set.

    """
    result = []

    # rows for group a
    for i in range(1, a+1):
        row = []
        neighbors = []

        # set of group b vertices
        # need to recalculate this for every row
        grp_b_verts = list(np.arange(a+1, a+b+1))

        # generate neighbors
        degree = np.random.choice(list(np.arange(1, b+1)))
        for j in range(degree):
            neighbor = np.random.choice(grp_b_verts)
            neighbors.append(neighbor)
            grp_b_verts.remove(neighbor)

        # construct first a rows
        for j in range(1, a+1):
            if j == i:
                row.append(float(degree) / 2)
            else:
                row.append(0)
        for j in range(a+1, a+b+1):
            if j in neighbors:
                row.append(-1)
            else:
                row.append(0)
        result.append(row)
```

```
    # degrees of group b vertices
    degrees = []
    for i in range(a+1, a+b+1):
        degree = 0
        for row in result:
            if row[i - 1] == -1:
                degree += 1
        degrees.append(degree)

    # rows for group b
    for i in range(a+1, a+b+1):
        row = [0] * a
        for j in range(a + 1, a+b+1):
            if j == i:
                row.append(float(degrees[j - a - 1])/2)
            else:
                row.append(0)
        result.append(row)

    result = sym_mat(result)

    return result
```

PBH Test

```
import numpy as np
import itertools


def control_set(n):
    """"
    Generates a list of all n-dimensional binary vectors
    expect for the all-ones vector and the zero vector.

    * n = dimension of matrix/vertices of graph.

    """
    binarylst = list(itertools.product([0, 1], repeat=n))
    B = []

    for i in range(pow(2,n)):
        B.append(list(binarylst[i]))

    B.remove(B[pow(2,n)-1])
    B.remove(B[0])
    return B

def correction(x):
    """
    The np.linalg.eig() function will often return extremely
    small floats instead of zero for eigenvalues/components
    of eigenvectors. The PBH test requires knowing exactly
    what of these are zero, so this functions serves to
    correct for this error.
```

```
    """
    if abs(x)<1e-10:
        return 0
    else:
        return x

def find_evecs(mat):
    """
    Returns an array whose entries are the eigenvectors of matrix "mat".

    """
    result = []

    evecs = np.linalg.eig(mat)[1]
    evecs = np.transpose(evecs)
    for i in range(n):
        result.append(list(map(correction,evecs[i])))

    return np.array(result)

# count function, accouting for rounding error
def correction_count(x,a):
    """
    This function count the number of entries of x in array
    a, with a tolerance to accont for numpy rounding errors.
    """
    result = []

    for num in a:
        if abs(num-x)<1e-14:
            result.append(x)

    return len(result)

def eig_degen_test(mat):
    """
    Checks if matrix "mat" has degenerate eigenvalues or not.

    """
    all_ones = [1] * n
    multiplicities = []
    evals = list(map(correction,np.linalg.eig(mat)[0]))

    for x in evals :
        multiplicities.append(correction_count(x,evals))

    if multiplicities == all_ones:
        return "nondegenerate"
    else:
        return "degenerate"

def PBH_test(mat):
    """
```

```
    Determines the controllability class of matrix "mat".

    """
    evecs = find_evecs(mat)
    zeros = []
    degen = eig_degen_test(mat)

    for control_vec in B:
        dot_prods = []

        for evec in evecs:
            dot_prod = sum(control_vec * evec)
            dot_prods.append(dot_prod)
        dot_prods = list(map(correction,dot_prods))

        if dot_prods.count(0)>0:
            zeros.append(0)

    if degen=="degenerate":
        return "completely uncontrollable"

    if len(zeros) == pow(2,n)-2:
        return "completely uncontrollable"

    if len(zeros) == 0:
        return "essentialy controllable"
    else:
        return "conditionally controllable"
```

Matrix Visualization

```
import networkx as nx
import matplotlib.pyplot as plt

def visualize_graph(matrix):
    """
    Draws a graph described by Laplacian
    matrix "mat" with unlabeled nodes.

    """
    g = nx.Graph()

    for i in range(n):
        for j in range(n):
            if matrix[i][j] == -1:
                g.add_edge(i,j)
    nx.draw(g)

def visualize_graph_num(matrix):
    """
    Draws a graph described by Laplacian
    matrix "mat" with labeled nodes (1-n).

    """
```

```
g = nx.Graph()

for i in range(n):
    for j in range(n):
        if matrix[i][j] == -1:
            g.add_edge(i+1,j+1)
nx.draw(g, with_labels = True)
```